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Combinatorics of lattice paths with and without spikes

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ABSTRACT

We derive a series of results on random walks on a d-dimensional hypercubic lattice (lattice paths). We introduce the notions of terse and simple paths corresponding to the path having no backtracking parts (spikes). These paths label equivalence classes which allow a rearrangement of the sum over paths. The basic combinatorial quantities of this construction are given. These formulas are useful when performing strong coupling (hopping parameter) expansions of lattice models. Some applications are described.

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1 Introduction

The strong coupling expansion is a useful analytical technique to study lattice models. In the context of lattice gauge theories it has been used since early days to investigate the behaviour of the system far from the continuum limit [1]. This technique is related to the high temperature expansions of classical Statistical Mechanics. In the case of matter fields (continuous spin variables) with nearest neighbor interactions the technique involves a *hopping parameter expansion* giving rise to a representation of the free energy and the propagators (correlation functions) in terms of random walks (see Ref. [2] and references therein). In its application to gauge theories the contribution of each random walk includes the corresponding Wilson loop. Since the early calculations of this type [3] the special behaviour of *backtracking* paths was recognized. Backtracking occurs when the walk makes two consecutive opposite steps: the first in one direction and the next one in the reverse direction. Part of their special character is related to the fact that the expectation value of a Wilson loop is suppressed like an exponential of its area. At infinitely strong coupling and large N this leaves loops which are *pure backtrackers*. Actually, for unitary gauge fields the backtracking part of a path is independent of the gauge fields themselves. The problem of summing over these type of paths becomes independent of the expectation value of gauge fields and is a pure combinatorial problem. The problem was avoided, nevertheless, in the mentioned lattice QCD strong coupling expansions [3, 4] by setting the Wilson parameter r equal to 1. This kills backtracking paths from quark propagators. Later, strong coupling calculations at $r \neq 1$ were performed by the effective potential method [5] and its connection to the backtracking-path resummation problem was never established. Recently,

motivated by the strong coupling expansion of supersymmetric Yang-Mills theory [6] we fell back into the problem. The effective potential method was unavailable in this case and we attacked the problem of backtracking random walks. The solution, presented here, gives for the lattice QCD case results in agreement with the effective potential method. We believe that the results and techniques can be of interest in other situations and hence we decided to collect proofs and results in this paper.

This paper is written in a self-contained form. In the next section we introduce the basic notation and definitions. Not to conflict with other definitions we will refer to random walks as *lattice paths* and to backtracking parts as *spikes*. Then, the main result on resummation over pure backtracking parts is presented. Section 3 gives the expression of a kind of matrix generating functional for paths with no spikes. This expression is useful for strong coupling expansions of lattice gauge theories. In Section 4, we consider closed paths. In this case the notion of a *simple* path turns out to be useful. Formulas similar to those given in the previous two sections are given for the case of simple paths. Finally, in section 5 we give a few applications which exemplify the way in which the previous results enter in physical expressions within strong coupling expansions. This includes the pure spike contribution to the free energy of a Gaussian model and the mesonic effective action. This is the surviving part if the fields are coupled to a random $U(N)$ gauge field at large N [7]. The reader who is not interested in proofs can jump directly to the last section. Our main results are given in formulas (14), (20), (32-34), (39), (40), (41) and (44).

2 Reducing paths

In this section we will introduce the basic notation and definitions. We will be working in arbitrary space-time dimension d . Vector indices μ go from 0 to $d-1$. We will need to introduce an index set I with $2d$ elements. For every space-time direction μ there are two elements μ and $\bar{\mu}$. They correspond to the two senses associated to each direction (forward and backward). Now consider our space-time lattice $\mathcal{L} \equiv \mathbf{Z}^d$. We might associate to any element α in I a lattice vector $V(\alpha)$ as follows:

$$\mu \longrightarrow V(\mu) \equiv e^{(\mu)} \quad \bar{\mu} \longrightarrow V(\bar{\mu}) \equiv -e^{(\mu)} ,$$

where $e^{(\mu)}$ is the unit vector in the μ direction. Given one element $\alpha \in I$ the element $\bar{\alpha}$ denotes the reversely oriented one ($\bar{\bar{\mu}} = \mu$).

Now we proceed to give a few definitions:

Definition 1 *A lattice path of length L is an element $\gamma \equiv (n, \vec{\alpha}) \in \mathcal{L} \times I^L$. The point $n \in \mathcal{L}$ is the **origin** of the path, and $\vec{\alpha}$ is the **path sequence**, specifying the steps to take to describe the path.*

The **endpoint** of a path $(n, \alpha_1, \dots, \alpha_L)$ is given by the lattice point $m = n + V(\vec{\alpha}) = n + \sum_{i=1}^L V(\alpha_i)$. We might now introduce the following nomenclature for the set of paths. Let $\mathcal{S}_L(n)$ be the space of all paths with origin n and length L . $\mathcal{S}(n)$ labels the set of all paths with origin n and any length. We might also fix the origin and endpoint and write $\mathcal{S}_L(n \rightarrow m)$.

The total number of paths of length L is easy to count: $N(L) = (2d)^L$. For length $L = 0$ we will consider that there is a unique path with origin in n , which we will call the **path of zero length**. To any path $\gamma \equiv (n, \vec{\alpha})$ of length L , there corresponds a path called its **reverse path** of equal length and labeled $\gamma^{-1} \equiv (m, \vec{\beta})$. The origin of the reverse path m is the endpoint of

the original path and vice versa. The path sequence is the reversely ordered one ($\beta_i = \bar{\alpha}_{L-i+1}$). We will also introduce a path composition operation. Given a path $\gamma \equiv (n, \vec{\alpha})$ whose endpoint is n , and another path $\gamma' \equiv (m, \vec{\beta})$, we can construct the composed path $\gamma \circ \gamma' \equiv (n, \vec{\alpha}, \vec{\beta})$.

Now we will give some more definitions.

Definition 2 *A path $\gamma \equiv (n, \alpha_1, \dots, \alpha_L)$ has **spikes** if there exist one integer i ($1 \leq i \leq (L-1)$) such that $\alpha_{i+1} = \bar{\alpha}_i$. In the converse case one says that the path is **terse** or has **no spikes**. The set of all paths without spikes (terse) of length L and origin n is labeled $\bar{\mathcal{S}}_L(n)$ ($\bar{\mathcal{S}}_L(n \rightarrow m)$ if the endpoint is fixed to m).*

It is not difficult to obtain $\bar{N}(L)$: the number of elements of $\bar{\mathcal{S}}_L(n)$. Its value is $2d(2d-1)^{L-1}$ for $L \geq 1$. The path of zero length is terse $\bar{N}(0) = 1$.

Now we will classify the set of paths into subsets labeled by a terse path. Let us first present the results:

- *There exist a projection $\pi : \mathcal{S}(n \rightarrow m) \rightarrow \bar{\mathcal{S}}(n \rightarrow m) \subset \mathcal{S}(n \rightarrow m)$ such that to every path γ it associates a terse path $\pi(\gamma)$, called its **reduced path**. If the length of γ is L , then the length of $\pi(\gamma)$ is $L - 2p$, for some integer p .*

Definition 3 *If $\pi(\gamma)$ is the path of length zero, then γ is said to be a **pure spike path**.*

The construction of $\pi(\gamma)$ proceeds iteratively. If the path $\gamma = (n, \vec{\alpha})$ is terse, then $\pi(\gamma) = \gamma$. Otherwise, one can start to scan the sequence of indices i in increasing order, until one finds a value of i such that $\alpha_{i+1} = \bar{\alpha}_i$. This by hypothesis must hold for some i . Then, one can eliminate the elements i and

$i + 1$ from the sequence, thus defining a new path of length $L - 2$. Then, one can apply the procedure once more to the resulting path. In this way, one must proceed iteratively until the iteration terminates. This must necessarily happen since the length of the original path L is finite. The iteration can terminate in two ways. Most frequently, one would reach, at some stage of the iterative procedure, a path without spikes. Then this is precisely $\pi(\gamma)$. In some cases, the iteration proceeds until there are no more elements left in the sequence $\vec{\alpha}$. In this case we would say that the corresponding reduced path is the path of zero length, and γ is a pure spike.

The sets $\pi^{-1}(\hat{\gamma})$ will play an important role in our construction. Our main interest is to determine the numbers $N(\hat{\gamma}, p)$: the number of paths of length $2p + \bar{L}$ whose reduced path is $\hat{\gamma}$ (whose length is \bar{L}). For the construction we will need to introduce two groups of operations on the sets of paths:

$$\phi : \mathcal{S}_L(n) \longrightarrow \mathcal{S}_{L-1}(n) \quad (1)$$

such that for $\gamma = (n, \alpha_1, \dots, \alpha_L)$, we have $\phi(\gamma) = (n, \alpha_1, \dots, \alpha_{L-1})$

$$\phi_\alpha : \mathcal{S}_L(n) \longrightarrow \mathcal{S}_{L+1}(n) \quad (2)$$

For $\gamma = (n, \alpha_1, \dots, \alpha_L)$ and $\alpha \in I$, we have $\phi_\alpha(\gamma) = (n, \alpha_1, \dots, \alpha_L, \alpha)$

What we need to know is what is the interplay between these operations and the projection π . Let us consider a path $\gamma = (n, \alpha_1, \dots, \alpha_L)$ whose reduced path is $\pi(\gamma) = (n, \beta_1, \dots, \beta_{\bar{L}})$. We are interested in the reduced path $\pi(\phi_\alpha(\gamma))$. By the iterative definition of π , we see that after some iterations we would end up with a path $\phi_\alpha(\pi(\gamma))$. Now there can be two cases: if $\bar{\alpha} \neq \beta_{\bar{L}}$ this path is terse and hence $\pi(\phi_\alpha(\gamma)) = (n, \beta_1, \dots, \beta_{\bar{L}}, \alpha) = \phi_\alpha(\pi(\gamma))$; for the special case $\bar{\alpha} = \beta_{\bar{L}}$, one must still apply one reduction step and the result is $\pi(\phi_{\beta_{\bar{L}}}(\gamma)) = (n, \beta_1, \dots, \beta_{\bar{L}-1}) = \phi(\pi(\gamma))$. Now we study $\pi(\phi(\gamma))$. There are again two cases: if $\alpha_L = \beta_{\bar{L}}$ the result is $(n, \beta_1, \dots, \beta_{\bar{L}-1}) = \phi(\pi(\gamma))$; in the

rest of cases we have $(n, \beta_1, \dots, \beta_{\bar{L}}, \bar{\alpha}_L) = \phi_{\bar{\alpha}_L}(\pi(\gamma))$. These results can be proven in a similar way as for ϕ_α .

Now we will make use of the previous results. Consider a terse path $\hat{\gamma} = (n, \beta_1, \dots, \beta_{\bar{L}})$ of non-zero length \bar{L} , and consider the set $\mathcal{S}(\hat{\gamma}, p)$ of all paths γ of length $L = \bar{L} + 2p$, with $p \geq 1$ an integer, whose reduced path is $\hat{\gamma}$. Then we can conclude:

- *The application ϕ induces a mapping from $\mathcal{S}(\hat{\gamma}, p)$ into $\mathcal{S}(\phi(\hat{\gamma}), p) \cup_{\alpha \neq \beta_{\bar{L}}} \mathcal{S}(\phi_{\bar{\alpha}}(\hat{\gamma}), p - 1)$, which is bijective.*
- *Henceforth, the number of paths $N(\hat{\gamma}, p)$ in $\mathcal{S}(\hat{\gamma}, p)$ satisfies:*

$$N(\hat{\gamma}, p) = N(\phi(\hat{\gamma}), p) + \sum_{\alpha \neq \beta_{\bar{L}}} N(\phi_{\bar{\alpha}}(\hat{\gamma}), p - 1) .$$

Actually, the number $N(\hat{\gamma}, p)$ does not depend on the path $\hat{\gamma}$ but only on its length \bar{L} . We thus conclude:

$$N(\bar{L}, p) = N(\bar{L} - 1, p) + (2d - 1) N(\bar{L} + 1, p - 1) . \quad (3)$$

- *For a pure spike path $(n, \alpha_1, \dots, \alpha_L)$, we might apply ϕ and produce a path of length $L - 1$ with reduced path $(n, \bar{\alpha}_L)$. This is also bijective and leads to:*

$$N(0, p) = 2d N(1, p - 1). \quad (4)$$

The proof of the previous statements is as follows. The bijectivity can be shown by the existence of an inverse transformation. This is basically ϕ_α with α chosen appropriately. To prove that $N(\hat{\gamma}, p)$ only depends on the length can be done by induction. Prove directly by construction that the statement is true for paths of short length (it is easy to solve the problem up to $L = 4$ for example). Then one assumes that the statement is verified

up to length $L = \bar{L} + 2p$ (for any p). Then one can use the formulas (3) and (4) to show that the statement is true for paths of length $L + 1$. Notice that if the right hand side does not depend on the actual terse paths but only on its lengths, and since these lengths only depend on the length of the reduced path $\hat{\gamma}$ of the left hand side, the result follows.

By repeated application of the relations Eq. (3) and (4), together with the initial condition $N(\bar{L}, 0) = 1$, one can obtain all the $N(\bar{L}, p)$ values. To exploit these relations we will introduce the following generating functions:

$$F(\bar{L}, z) = \sum_{p=0}^{\infty} z^p N(\bar{L}, p) \quad (5)$$

$$G(y, z) = \sum_{\bar{L}=0}^{\infty} F(\bar{L}, z) y^{\bar{L}} \quad (6)$$

Multiplying the relation (3) by the appropriate powers of z and y and summing over p and \bar{L} , one gets:

$$G(y, z) - F(0, z) = y G(y, z) + \frac{(2d-1)z}{y} (G(y, z) - F(0, z) - yF(1, z)) \quad (7)$$

and from it, one can write G in terms of F :

$$G(y, z) = \frac{1}{(y(1-y) - (2d-1)z)} \left(\left(\frac{y}{2d} - (2d-1)z \right) F(0, z) + y \frac{2d-1}{d} \right) \quad (8)$$

Notice that the zeroes of the denominator in the previous expression can give rise to singularities, even for small values of z and y , unless the numerator vanishes at these zeroes. This must actually happen since G and F can be shown to be analytic in a neighborhood of $y = z = 0$ (This follows from $N(\bar{L}, p) < (2d)^{\bar{L}+2p}$). This allows one to determine $F(0, z)$:

$$F(0, z) = \frac{2d-1}{d} \frac{1}{1 + \frac{d}{d-1} \sqrt{1 - 4(2d-1)z}}. \quad (9)$$

Now plugging this expression into (8) we obtain the formula for $G(y, z)$.

The expressions can be simplified with a suitable change of variables. Let us introduce the variable ξ :

$$\xi(z) = \frac{1 - \sqrt{1 - 4(2d-1)z}}{2} \quad (10)$$

$$\text{with inverse } z(\xi) = \frac{\xi(1-\xi)}{2d-1} \quad (11)$$

Then one can conclude:

$$F(0, z(\xi)) = \frac{1}{1 - \frac{2d}{2d-1}\xi} \quad (12)$$

$$G(y, z(\xi)) = -\frac{(2d-1)(1-\xi)}{2d-1-2d\xi} \frac{1}{y+\xi-1} \quad (13)$$

Notice that the only dependence on y sits in the last denominator. It is now fairly simple to obtain $F(\bar{L}, z)$ by picking the relevant power of y in the expansion. One gets:

$$F(\bar{L}, z(\xi)) = \frac{1}{(1 - \frac{2d}{2d-1}\xi)} \frac{1}{(1-\xi)^{\bar{L}}} \quad (14)$$

The last formula is the main one of this chapter. From it one can obtain the numbers $N(\bar{L}, p)$, by differentiation or Cauchy integration. This we will do later.

Before, as a check of our formulas, one can compute the number of paths of length L as a sum over the number of terse paths times the number of paths of length L having a given terse path as reduced path:

$$(2d)^L = N(L) = \sum_{p=0}^{[\frac{L}{2}]} \bar{N}(L-2p) N(L-2p, p) \quad (15)$$

To check all formulas at the same time we can multiply the expression by z^L and sum over L . We get:

$$\frac{1}{1-2dz} = F(0, z^2) + \sum_{\bar{L}=1}^{\infty} z^{\bar{L}} F(\bar{L}, z^2) 2d (2d-1)^{\bar{L}-1} =$$

$$\frac{1}{(1 - \frac{2d}{2d-1}\xi(z^2))} (1 + \frac{2dz}{1 - \xi(z^2) - (2d-1)z}) = \frac{\frac{1 - \xi(z^2) + z}{(1 - \frac{2d}{2d-1}\xi(z^2))(1 - \xi(z^2) - (2d-1)z)}}{.} \quad (16)$$

For the third identity of the previous formula we have resummed a geometric series. Finally, to prove that the right hand side of the previous equation coincides with the left hand side, one must simply manipulate algebraically the expression and use the relation $\xi(z^2)(1 - \xi(z^2)) = (2d - 1)z^2$.

We conclude this section by extracting the numbers $N(L, p)$ themselves. This can be done by employing the expression of the generating function (14), and making a contour integral in the complex plane of z around the origin, and using Cauchy's theorem. It is more practical to change variables from z to ξ in the integral. Notice that for $|z|$ small enough, the contour in ξ also encircles the origin and the function $F(L, z(\xi))$ has no singularities inside. The resulting integrand is a product of negative powers of ξ and of $(1 - \xi)$, times the factor $1/(1 - \frac{2d}{2d-1}\xi)$ coming from (14). If one expands this denominator in powers of ξ , it is not hard to show that:

$$N(L, p) = \sum_{j=0}^p (2d)^j (2d - 1)^{p-j} \frac{(L + 2p - j - 1)!}{(p - j)! (L + p)!} (L + j) . \quad (17)$$

We see that the resulting expression is a polynomial in d of degree p . To obtain the coefficients of the different powers of d , one could expand the power of $(2d - 1)$ in powers of d and rearrange the summation. The method can be carried out but is fairly lengthy and complicated. A short cut to arrive to the same final expression is to multiply and divide Eq. (17) by $j!$. Then one replaces the factors $j!$ and $(L + 2p - j - 1)!$ by their standard integral representation(that of Euler's gamma function) and performs the sum in j . The expression is then given as a double integral over two variables α and β going from 0 to ∞ . Now one can perform the standard trick in computing

Feynman integrals by changing variables to $\lambda \equiv (\alpha + \beta)$ and the *Feynman parameter* $x \equiv \frac{\alpha}{\lambda}$. The integration over λ can be performed and we arrive at:

$$\begin{aligned} N(L, p) &= \binom{L+2p}{p} \int_0^1 dx x^{L+p-1} (2d-x)^{p-1} (L(2d-x) + 2dp(1-x)) = \\ &= \binom{L+2p}{p} \sum_{s=0}^p \frac{(2d)^s (-1)^{p-s}}{(L+2p-s)} \binom{p}{s} \left(L + \frac{s}{L+2p-s+1} \right) . \end{aligned} \quad (18)$$

The first equality in the previous expression is a Feynman parameter integral representation of the numbers $N(L, p)$. The second one is a representation as a polynomial in d , and it can be obtained easily from the other. We have given the expression for $N(L, p)$ for completeness, though in actual applications it is more useful to work with the generating function $F(L, z)$.

3 Summing over terse paths

In this section we will compute a matrix generating function for the set of terse paths $\bar{\mathcal{S}}_L(n)$. This generating function turns out to be useful in applications to strong coupling expansions of lattice models. Let us introduce a collection of matrices \mathbf{A}_α for $\alpha \in I$. The interesting quantity to study is:

$$\mathcal{T}(\mathbf{A}) = \sum_{L=0}^{\infty} \sum_{(n, \vec{\alpha}) \in \bar{\mathcal{S}}_L(n)} \mathbf{A}_{\alpha_1} \cdots \mathbf{A}_{\alpha_L}. \quad (19)$$

We will first compute $\mathcal{T}(\mathbf{A})$ for matrices \mathbf{A} satisfying $\mathbf{A}_\alpha \mathbf{A}_{\bar{\alpha}} = \lambda \mathbf{I}$ (*i.e.* a multiple of the identity). This condition is satisfied in some of the most important applications of the formula. At the end of the section we will give the more general formulas.

To facilitate the reading for those who are mostly interested in the result,

we will begin by giving the answer:

$$\mathcal{T}(\mathbf{A}) = (1 - \lambda)(1 + (2d - 1)\lambda - \tilde{\mathbf{A}})^{-1} \quad (20)$$

$$\text{with } \tilde{\mathbf{A}} = \sum_{\alpha \in I} \mathbf{A}_\alpha \quad (21)$$

The conditions on the matrices \mathbf{A} for which the previous expression applies can be read out from it. The eigenvalues of $\tilde{\mathbf{A}}$ must be small enough for the inverse matrix entering in Eq. (20) to exist. In the following paragraphs we will give the proof of this result.

We begin by considering the set of all terse paths, with origin in n , length $L \geq 1$ and ending with step α : $\bar{\mathcal{S}}_L^\alpha(n)$. Applying a similar definition as Eq. (19) to this set we get:

$$\mathcal{T}_\alpha(L, \mathbf{A}) = \sum_{\gamma \in \bar{\mathcal{S}}_L^\alpha(n)} \mathbf{A}_{\alpha_1} \cdots \mathbf{A}_\alpha. \quad (22)$$

Now, clearly the path $\phi(\gamma)$ is also a terse path and has length $L - 1$, but it cannot end with step $\bar{\alpha}$. Hence:

$$\mathcal{T}_\alpha(L + 1, \mathbf{A}) = \sum_{\beta \neq \bar{\alpha}} \mathcal{T}_\beta(L, \mathbf{A}) \mathbf{A}_\alpha. \quad (23)$$

The formula is valid for $L \geq 1$.

Now from it we will derive an equation for $\mathcal{T}_\alpha(\mathbf{A}) \equiv \sum_{L=1}^{\infty} \mathcal{T}_\alpha(L, \mathbf{A})$. Then our main quantity $\mathcal{T}(\mathbf{A})$ is given by $\mathbf{I} + \sum_{\alpha \in I} \mathcal{T}_\alpha(\mathbf{A})$. Summing Eq. (23) over L one gets:

$$\mathcal{T}_\alpha(\mathbf{A}) = (\mathcal{T}(\mathbf{A}) - \mathcal{T}_{\bar{\alpha}}(\mathbf{A})) \mathbf{A}_\alpha \quad (24)$$

Given $\mathcal{T}(\mathbf{A})$, these are coupled equations for the indices α and $\bar{\alpha}$. We then write them as a single vector equation:

$$(\mathcal{T}_\alpha(\mathbf{A}), \mathcal{T}_{\bar{\alpha}}(\mathbf{A})) \mathcal{H} = \mathcal{T}(\mathbf{A}) (\mathbf{A}_\alpha, \mathbf{A}_{\bar{\alpha}}) \quad (25)$$

where \mathcal{H} is an invertible matrix. This matrix and its inverse are given by the formulas:

$$\mathcal{H} = \begin{pmatrix} \mathbf{I} & \mathbf{A}_{\bar{\alpha}} \\ \mathbf{A}_\alpha & \mathbf{I} \end{pmatrix} \quad (26)$$

$$\mathcal{H}^{-1} = \frac{1}{(1-\lambda)} \begin{pmatrix} \mathbf{I} & -\mathbf{A}_{\bar{\alpha}} \\ -\mathbf{A}_\alpha & \mathbf{I} \end{pmatrix} \quad (27)$$

Then, for fixed $\mathcal{T}(\mathbf{A})$, one can solve for $\mathcal{T}_\alpha(\mathbf{A})$, obtaining:

$$\mathcal{T}_\alpha(\mathbf{A}) = \frac{\mathcal{T}(\mathbf{A})}{(1-\lambda)} (\mathbf{A}_\alpha - \lambda) . \quad (28)$$

Finally summing both sides of the equation over α we get:

$$\mathcal{T}(\mathbf{A}) - \mathbf{I} = \frac{\mathcal{T}(\mathbf{A})}{(1-\lambda)} (\tilde{\mathbf{A}} - 2d\lambda) . \quad (29)$$

From this equation one can solve for $\mathcal{T}(\mathbf{A})$ obtaining the formula (20).

We can now, as in the previous section, check the formula by using it in deriving a known result. Consider the sum over all paths $\gamma = (n, \vec{\alpha})$ of the ordered product of the matrices \mathbf{A}_α . Since this is a geometric series it is easily resummed to $(\mathbf{I} - \tilde{\mathbf{A}})^{-1}$. Now this result has to be reobtained by splitting the sum over paths into a sum over terse paths and a sum over paths whose reduced path is a given terse path. In this way one is making use of the results of the last section and of this section at the same time. The last summation can be performed in terms of the generating function studied in the last section. One has:

$$\sum_{L=0}^{\infty} \sum_{(n, \vec{\alpha}) \in \mathcal{S}_L(n)} \mathbf{A}_{\alpha_1} \cdots \mathbf{A}_{\alpha_L} F(L, \lambda) . \quad (30)$$

Now, given the form of $F(L, \lambda)$ given in expr. (14), one recognizes the structure given in Eq. (19) with the \mathbf{A}_α rescaled. We get:

$$\frac{1}{(1 - \frac{2d}{2d-1} \xi(\lambda))} \mathcal{T}(\mathbf{A}/(1 - \xi(\lambda))) \quad (31)$$

Now using our expression for $\mathcal{T}(\mathbf{A})$ (Eq. (20)) and the relation between $\xi(\lambda)$ and λ (Eq. 10) one obtains the known result.

Now, as announced at the beginning of the section we will give the result without imposing the condition $\mathbf{A}_\alpha \mathbf{A}_{\bar{\alpha}} = \lambda \mathbf{I}$. It is not difficult by following the same steps as before to show that in the general case we have:

$$\mathcal{T}(\mathbf{A}) = (1 - \tilde{\mathbf{B}})^{-1} \quad (32)$$

$$\mathcal{T}_\alpha(\mathbf{A}) = \mathcal{T}(\mathbf{A}) (\mathbf{A}_\alpha - \mathbf{A}_{\bar{\alpha}} \mathbf{A}_\alpha) (\mathbf{I} - \mathbf{A}_{\bar{\alpha}} \mathbf{A}_\alpha)^{-1} \quad (33)$$

$$\text{where } \tilde{\mathbf{B}} = \sum_\alpha (\mathbf{A}_\alpha - \mathbf{A}_{\bar{\alpha}} \mathbf{A}_\alpha) (\mathbf{I} - \mathbf{A}_{\bar{\alpha}} \mathbf{A}_\alpha)^{-1} \quad (34)$$

The previously given formula (20) follows as a special case from this one.

To conclude this section, we comment that, using the above formulas, one can derive matrix generating functions for the sum of terse paths with fixed origin and endpoint. For that purpose one simply has to multiply \mathbf{A}_α by a phase $e^{i\varphi_\alpha}$. For the reverse direction $\bar{\alpha}$ one has the complex conjugate phase ($\varphi_{\bar{\alpha}} = -\varphi_\alpha$), so that the condition $\mathbf{A}_\alpha \mathbf{A}_{\bar{\alpha}} = \lambda \mathbf{I}$ is respected. With a suitable integration over the phases φ_μ one can restrict the sum over terse paths to those having fixed endpoint. For example, if one wants to evaluate the contribution to $\mathcal{T}(\mathbf{A})$ from paths whose origin is n and endpoint is m one can simply write:

$$\prod_\mu \left(\int_0^{2\pi} \frac{d\varphi_\mu}{2\pi} e^{i\varphi_\mu(n_\mu - m_\mu)} \right) \mathcal{T}(e^{i\varphi_\alpha} \mathbf{A}) \quad . \quad (35)$$

4 Closed and Simple paths

In this section we will look at closed paths: a path such that its origin and endpoint coincide. One frequently encounters situations in which closed paths play an important role, such as in computing the free energy or the

fermion determinant. In those cases, for every such path one has to evaluate a trace. For that purpose, the notion of terse paths is somehow insufficient. We would like to single out those paths $(n, \vec{\alpha})$ for which the last step α_L differs from $\bar{\alpha}_1$. We will call those paths simple. Let us now give more precisely the definition.

Definition 4 *A path $\gamma = (n, \vec{\alpha})$ of length L is **simple**, if it is terse (without spikes) and in addition one has $\alpha_L \neq \bar{\alpha}_1$.*

Simple paths can be open or closed, however their usefulness appears normally when they are closed. The set of simple paths of length L and origin in n is labeled $\tilde{\mathcal{S}}_L(n)$. The set $\tilde{\mathcal{S}}_0(n)$ is given by the path of zero length.

By a similar procedure to the one followed in section 1, one can associate to any path γ a given simple path $\tilde{\pi}(\gamma)$. To construct $\tilde{\pi}(\gamma)$ one starts by obtaining the reduced path $(\pi(\gamma) = \hat{\gamma} \equiv (n, \vec{\beta}))$ associated to γ . Let us consider that its length is L and its origin n . If this terse path is simple, then this is precisely $\tilde{\pi}(\gamma)$. If not, it is due to β_L being equal to $\bar{\beta}_1$. Hence, we eliminate the first and last steps in the sequence $(\beta_1 \text{ and } \beta_L)$. The resulting path is terse and has length $L - 2$. Notice, however, that its origin is now $n + V(\beta_1)$, and not n . If the path is simple, then it coincides with $\tilde{\pi}(\gamma)$, otherwise one has to repeat the procedure once more. Eventually, one reaches a simple path, which could be just a path of zero length.

Our first goal is to develop similar counting rules for simple paths as those obtained in section 2 for terse paths. In particular we are interested in computing the numbers $\tilde{N}(\tilde{\gamma}, p)$: the number of paths γ of length $l+2p$ whose associated simple path is $\tilde{\pi}(\gamma) = \tilde{\gamma}$ (of length l). The generating function of these numbers is:

$$\tilde{F}(\tilde{\gamma}, z) = \sum_{p=0}^{\infty} z^p \tilde{N}(\tilde{\gamma}, p) \quad . \quad (36)$$

In what follows we will compute this generating function and the numbers $\tilde{N}(\tilde{\gamma}, p)$.

The procedure that we will employ is to relate these numbers with $N(l, p)$. For that purpose consider a path γ with $\tilde{\pi}(\gamma) = \tilde{\gamma} \equiv (m, \vec{\alpha})$ and consider its reduced path $\pi(\gamma) \equiv (n, \vec{\beta})$. It is clear from the description of the construction of $\tilde{\pi}(\gamma)$ that the path $\pi(\gamma)$ must be the composition of three paths:

$$\pi(\gamma) = s \circ \tilde{\gamma} \circ s^{-1} .$$

The path $s \equiv (n, \vec{\rho})$ is a terse path of length p' ($0 \leq p' \leq p$) going from n to m . If the path $\tilde{\gamma}$ has length l then $\rho_{p'} \neq \bar{\alpha}_1, \alpha_l$. This must hold, since the composition $s \circ \tilde{\gamma} \circ s^{-1}$ must be terse. Conversely all paths γ having $\pi(\gamma) = s \circ \tilde{\gamma} \circ s^{-1}$ have $\tilde{\gamma}$ as its associated simple path. Hence, all one needs to do is to count for each case of $\pi(\gamma)$ the number of paths γ of length $l + 2p$.

The main formula is:

$$\tilde{N}(\tilde{\gamma}, p) = N(l, p) + \sum_{p'=1}^p N(l + 2p', p - p') (2d - 2)(2d - 1)^{p'-1} \quad (37)$$

The quantity $(2d - 2)(2d - 1)^{p'-1}$ counts the number of acceptable terse paths s of length p' going from any point n to m . The word acceptable refers to the condition $\rho_{p'} \neq \bar{\alpha}_1, \alpha_l$. The previous formula (37) is valid for l and p strictly positive. If any of the two is zero then $\tilde{N} = N$.

A first conclusion from formula (37) is that $\tilde{N}(\tilde{\gamma}, p)$ only depends on the length l of the simple path $\tilde{\gamma}$. If we multiply both sides of equation by z^p and sum over p , we get (for $l > 0$):

$$\tilde{F}(l, z) = F(l, z) + \sum_{p'=1}^{\infty} (2d - 2)(2d - 1)^{p'-1} z^{p'} \quad (38)$$

Now using the form of $F(l, z)$ one gets:

$$\tilde{F}(l, z) = \frac{1}{1 - 2\xi(z)} \frac{1}{(1 - \xi(z))^l} . \quad (39)$$

This formula is valid for $l > 0$. This is complemented by $\tilde{F}(0, z) = F(0, z)$. To extract from $\tilde{F}(l, z)$ the numbers $\tilde{N}(l, p)$, one proceeds as before by Cauchy integration. The calculation is now much simpler since $(1 - 2\xi(z))$ is up to a constant the jacobian for the change of variables from z to ξ . Finally one gets ($l > 0$):

$$\tilde{N}(l, p) = \frac{(2d - 1)^p (2p + l)!}{p! (p + l)!} . \quad (40)$$

In some applications, one is interested in a slight variant of the generating function $\tilde{F}(l, z)$. Its definition and final expression is given by:

$$\widetilde{F}'(l, z) \equiv \sum_{p=0}^{\infty} \frac{1}{l + 2p} \tilde{N}(l, p) z^p = \frac{1}{l(1 - \xi(z))^l} \quad (41)$$

The last expression, valid for positive l , could be obtained after some work by integration of $\tilde{F}(l, z)$. The $\frac{1}{l+2p}$ in the definition of \widetilde{F}' occurs naturally when the sum of closed paths is the result of a fermionic or bosonic determinant. We complement this result with the one for $l = 0$:

$$\widetilde{F}'(0, z) \equiv \sum_{p=0}^{\infty} \frac{1}{2p} N(0, p) z^p = d \log(1 - \xi(z)) + (d - 1) \log(1 - \frac{2d}{2d - 1} \xi(z)) . \quad (42)$$

The remaining part of this section is dedicated to the evaluation of sums over simple paths. The basic quantity is:

$$\tilde{\mathcal{T}}(\mathbf{A}) = \sum_{\bar{L}=0}^{\infty} \sum_{(n, \vec{\alpha}) \in \tilde{\mathcal{S}}_L(n)} \mathbf{A}_{\alpha_1} \cdots \mathbf{A}_{\alpha_{\bar{L}}} . \quad (43)$$

where $\tilde{\mathcal{S}}_L(n)$ is the set of all simple paths with origin in n and length L . In short, what we want is the generalization of the quantity defined in Eq. (19) but restricted to simple closed paths. Similarly to what we did in Section 3, we will first present the result, and then give the derivation. We obtain:

$$\tilde{\mathcal{T}}(\mathbf{A}) = \frac{1}{(1 - \lambda)} (2\lambda(d - 1) + (1 + \lambda^2(2d - 1))\mathbf{H} - \sum_{\alpha \in I} \mathbf{A}_{\alpha} \mathbf{H} \mathbf{A}_{\bar{\alpha}}) \quad (44)$$

$$\text{with } \mathbf{H} = (1 + (2d - 1)\lambda - \tilde{\mathbf{A}})^{-1} \quad (45)$$

where $\mathbf{A}_\alpha \mathbf{A}_{\bar{\alpha}} = \lambda \mathbf{I}$ as in Section 3.

The derivation follows a similar track to the one employed for $\mathcal{T}(\mathbf{A})$. Our first goal is the calculation of the quantity $\mathcal{T}_{\alpha\alpha'}(L, \mathbf{A})$ given by:

$$\mathcal{T}_{\alpha\alpha'}(L, \mathbf{A}) = \sum_{(n, \vec{\alpha}) \in \tilde{\mathcal{S}}_L^{\alpha\alpha'}(n)} \mathbf{A}_{\alpha_1} \cdots \mathbf{A}_{\alpha_L} , \quad (46)$$

where $\tilde{\mathcal{S}}_L^{\alpha\alpha'}(n)$ is the set of all simple paths $(n, \vec{\alpha})$ of length L ($L > 2$) and origin in n such that $\alpha_1 = \alpha$ and $\alpha_L = \alpha'$. The main iteration equation allowing the evaluation of this quantity is:

$$\mathcal{T}_{\alpha\alpha'}(L+2, \mathbf{A}) = \sum_{\alpha \neq \beta; \alpha' \neq \bar{\beta}'} \mathbf{A}_\alpha \mathcal{T}_{\beta\beta'}(L, \mathbf{A}) \mathbf{A}_{\alpha'} \quad (47)$$

The sum of $\mathcal{T}_{\alpha\alpha'}(L, \mathbf{A})$ over L ranging from 2 to ∞ is denoted $\mathcal{T}_{\alpha\alpha'}(\mathbf{A})$. An equation for this quantity follows from summing both sides of Eq. (47) over L . After similar manipulations as those of Section 3, one gets:

$$\mathcal{T}_{\alpha\alpha'}(\mathbf{A}) = \mathbf{A}_\alpha \mathcal{T}_{\bar{\alpha}\bar{\alpha}'}(\mathbf{A}) \mathbf{A}_{\alpha'} + S_{\alpha\alpha'} \quad (48)$$

$$\text{with } S_{\alpha\alpha'} = -\delta_{\alpha\bar{\alpha}'}\lambda + \lambda\delta_{\alpha\alpha'}\mathbf{A}_\alpha + (1 + \lambda)\mathbf{A}_\alpha \mathbf{H} \mathbf{A}_{\alpha'} + \lambda(\mathbf{H} \mathbf{A}_{\alpha'} + \mathbf{A}_\alpha \mathbf{H})$$

where \mathbf{H} is the quantity defined in Eq. (45). Finally combining the equation for $\mathcal{T}_{\alpha\alpha'}(\mathbf{A})$ and for $\mathcal{T}_{\bar{\alpha}\bar{\alpha}'}(\mathbf{A})$, one can solve for $\mathcal{T}_{\alpha\alpha'}(\mathbf{A})$:

$$\mathcal{T}_{\alpha\alpha'}(\mathbf{A}) = \frac{1}{(1 - \lambda)} (\lambda(-\delta_{\alpha\bar{\alpha}'} + \lambda\delta_{\alpha\alpha'}\mathbf{A}_\alpha - \mathbf{H} \mathbf{A}_{\alpha'} - \mathbf{A}_\alpha \mathbf{H}) + \mathbf{A}_\alpha \mathbf{H} \mathbf{A}_{\alpha'} + \lambda^2 \mathbf{H}) \quad (49)$$

The previous quantity can be related to $\tilde{\mathcal{T}}(\mathbf{A})$ as follows:

$$\tilde{\mathcal{T}}(\mathbf{A}) = \mathbf{I} + \tilde{\mathbf{A}} + \sum_{\alpha \neq \bar{\alpha}'} \mathcal{T}_{\alpha\alpha'}(\mathbf{A}) . \quad (50)$$

Using this result in combination with Eq. (49) one obtains the final formula (44). One can again check the validity of the expression by using it in

reobtaining a known result. We leave this to the reader. We recall that in the definition of $\tilde{T}(\mathbf{A})$ one sums over all simple paths, closed or open. Restricting oneself to closed paths can be done with the same technique explained at the end of the last section.

5 Discussion

In this section we will exemplify how to apply the previous results to some physical problems. We consider a lattice model involving continuous spin variables with nearest neighbor interactions. These lattice fields can be real or complex valued or Grassman variables if they describe fermions. To apply the path representation we need a quadratic action or Hamiltonian in these fields. For example, for complex fields one has:

$$\sum_i \phi^a(n)^\dagger \phi^b(m) M^{ab}(n, m) \quad . \quad (51)$$

The indices n, m label lattice points and the indices a, b are internal. The matrix M will depend on other fields. For instance, in many cases constraints or non-quadratic terms in the lattice action can be rewritten as a quadratic (gaussian) Hamiltonian with the aid of auxiliary fields. Then, one can integrate out these complex fields ($\phi^a(n)$) using the Gaussian integration formulas. The two quantities entering the final expressions are the determinant of M ($\det M$) and the inverse of M . Now, the nearest neighbor character of our matrix manifests itself in that we can write (after an adequate re-scaling of the fields if necessary):

$$M = \mathbf{I} - \sum_{\alpha \in I} \Delta_\alpha \quad , \quad (52)$$

where the matrix Δ_α can be written as:

$$\Delta_\alpha^{ab}(n, m) = \mathbf{A}_\alpha^{ab}(n) \delta_{m n+V(\alpha)} . \quad (53)$$

It only produces transitions between a lattice point n and its neighbor in the α direction $n + V(\alpha)$. It is this form of the matrix M what allows a random walk (lattice path) representation of the determinant or the inverse of M . Our formulas allow a rearrangement of this sum over paths into a sum over simple closed paths or terse paths respectively. This is feasible whenever:

$$\Delta_\alpha \Delta_{\bar{\alpha}} = \Lambda \quad (54)$$

with Λ a matrix which is independent of the lattice point. This occurs naturally whenever the matrix Δ_α , although dependent on the lattice point, involves unitary link fields like in $U(N)$ or Z_N gauge theories. We will restrict to the case when Λ is a multiple of the identity $\lambda \mathbf{I}$.

Now if we denote by $\mathbf{A}(\gamma)$ the ordered product of the matrices $\mathbf{A}_\alpha(n)$ along the path γ , we can write:

$$\log(\det(M))/\mathcal{V} = \left(-d \log(1 - \xi(\lambda)) + (d-1) \log(1 - \frac{2d}{2d-1} \xi(\lambda)) \right) \text{Tr}(\mathbf{I}) + \frac{1}{\mathcal{V}} \sum_{n \in \mathcal{L}} \sum_{l=1}^{\infty} \frac{1}{l} \sum_{\tilde{\gamma} \in \tilde{\mathcal{S}}_l} (n \rightarrow n) \frac{\text{Tr}(A(\tilde{\gamma}))}{(1-\xi)^l} , \quad (55)$$

where \mathcal{V} is the lattice volume and $\xi(\lambda)$ is defined in Eq. 10. To arrive to the previous equation, we have rearranged as usual the sum over paths into a sum over simple paths, and used the results of the previous sections. The term proportional to $\text{Tr}(\mathbf{I})$, equal to $\tilde{F}'(0, \lambda)$, gives the contribution of pure spike paths. In some theories, like $U(N)$ gauge theories at strong coupling in the large N limit with either bosonic or fermionic spin fields in either the fundamental or adjoint representation, this term turns out to be the only surviving one [7]. Thus, up to a multiplicative constant depending on the

type of field, $\tilde{F}'(0, \lambda)$ (Eq. (42)) gives the free energy per unit volume in that limit. We suggest that in other theories, the rearrangement into simple paths could be an effective method to perform the summation over paths.

Now, as an additional application, let us compute the pure spike contribution to the *mesonic effective potential*. Let us add to the action (51) a mesonic source term:

$$-\sum_{n,a,b} \phi^a(n)^\dagger \phi^b(n) J^{ab}(n) \quad . \quad (56)$$

where $J(n)$ acts as the source of local field bilinears (mesons). Integrating over the gaussian fields ϕ we get the connected generating functional $W(J)$:

$$W(J) \equiv \log(Z(J)/Z(0)) = \sum_{k=1}^{\infty} \frac{1}{k} \text{Tr}((M^{-1}J)^k) \quad , \quad (57)$$

where the trace includes a summation over lattice points. Each factor of M^{-1} can be expanded into a sum over paths (random walks). The pure spike contribution $W_0(J)$ is that in which the overall path obtained within each trace is a pure spike path. Again this contribution would be the leading one if the matrices $\mathbf{A}_\alpha(n)$ entering Eq. (51) involve random $U(N)$ fields at large N . In the subsequent expressions only the remaining part of the \mathbf{A}_α would enter, which we will assume to be independent of the lattice point in what follows. In order to implement the restriction to pure spike paths, it is convenient to express the propagators M^{-1} as a sum over terse paths:

$$(M^{-1}(n, m))^{ab} = \frac{1}{(1 - \frac{2d}{2d-1}\xi)} \sum_{l=0}^{\infty} \sum_{\hat{\gamma} \in \mathcal{S}_l(n \rightarrow m)} \frac{(A(\hat{\gamma}))^{ab}}{(1 - \xi)^l} \quad . \quad (58)$$

We then obtain for $W_0(J)$:

$$\begin{aligned} W_0(J) = & \sum_{k=1}^{\infty} \frac{1}{k} \sum_{x_1, \dots, x_k \in \mathcal{L}} \sum_{\hat{\gamma}_1 \in \bar{\mathcal{S}}(x_1 \rightarrow x_2)} \cdots \sum_{\hat{\gamma}_k \in \bar{\mathcal{S}}(x_k \rightarrow x_1)} \\ & \text{Tr}(J'(x_1) A'(\hat{\gamma}_1) J'(x_2) \cdots A'(\hat{\gamma}_k)) \Theta(\hat{\gamma}_1 \circ \hat{\gamma}_2 \cdots \circ \hat{\gamma}_k) \end{aligned} \quad (59)$$

where $\Theta(\gamma)$ is 1 if γ is a pure spike path and zero otherwise, and the rescaled quantities \mathbf{A}', J' are given by:

$$\mathbf{A}'_\alpha = \frac{\mathbf{A}_\alpha}{(1-\xi(\lambda))} \quad (60)$$

$$J'(n) = \frac{J(n)}{(1-\frac{2d}{2d-1}\xi(\lambda))} \quad . \quad (61)$$

We see that the net effect of replacing the sum over paths by a sum over terse paths is precisely this re-scaling, as follows from our results of section 2. The first two terms of $W_0(J)$ are:

$$W_0(J) = \sum_{x \in \mathcal{L}} \text{Tr}(J'(x)) + \frac{1}{2} \sum_{x_1, x_2 \in \mathcal{L}} \overline{J}'(x_1) \mathcal{P}(x_1 \rightarrow x_2) J'(x_2) + \dots \quad (62)$$

The linear term in J' is trivial since the only path that contributes is the path of zero length. The constraint $\Theta(\hat{\gamma}_1 \circ \hat{\gamma}_2)$ for the quadratic term implies that $\hat{\gamma}_2$ must be the reverse path of $\hat{\gamma}_1$. The resummation over terse paths can be done with the help of the formulas of section 3. One obtains the following explicit expression of the propagator $\mathcal{P}(x_1 \rightarrow x_2)$:

$$\mathcal{P}(x_1 \rightarrow x_2) = \prod_\mu \left(\int_0^{2\pi} \frac{d\varphi_\mu}{2\pi} e^{i\varphi_\mu(x_1 - x_2)_\mu} \right) (1 - \lambda') (1 + (2d-1)\lambda' - \mathbf{B})^{-1} \quad (63)$$

where:

$$\lambda' = \frac{\lambda^2}{(1-\xi(\lambda))^4} \quad (64)$$

$$\mathbf{B} = \sum_{\alpha \in I} e^{i\varphi_\alpha} \mathbf{A}'_\alpha \otimes (\mathbf{A}'_{\bar{\alpha}})^t \quad (65)$$

These expressions were used in our recent paper on $N = 1$ SUSY Yang-Mills [6]. In formula (62) $J'(n)$ has to be looked at as a column vector on which the matrix \mathcal{P} acts. Then, $\overline{J}'(x_1)$ is the row vector whose elements are the transpose of J' .

Finally, we will address the calculation of the cubic term in $W_0(J)$. For that purpose we have to solve the constraint imposed by $\Theta(\hat{\gamma}_1 \circ \hat{\gamma}_2 \circ \hat{\gamma}_3)$. In the generic case, it can be solved as follows:

$$\hat{\gamma}_1 = s_2^\alpha \circ (s_3^\beta)^{-1} \quad (66)$$

$$\hat{\gamma}_2 = s_3^\beta \circ (s_1^\gamma)^{-1} \quad (67)$$

$$\hat{\gamma}_3 = s_1^\gamma \circ (s_2^\alpha)^{-1} \quad (68)$$

where $s_1^\gamma \in \bar{\mathcal{S}}^\gamma$ is a terse path ending with a step in the γ direction, and similar definitions apply for s_2 and s_3 . Furthermore, one must have $\alpha \neq \beta \neq \gamma \neq \alpha$. The exceptional cases occur when any of the paths s_i is a path of zero length. It is clear that the summation over the paths s_i can be done with the aid of the formulas of section 3. The best way to express the result is in terms of the mean mesonic field $\Phi^{ab}(x)$:

$$\Phi(x) = \sum_{x' \in \mathcal{L}} \mathcal{P}(x \rightarrow x') J'(x') \quad (69)$$

Then the cubic term in $W_0(J)$ becomes:

$$\sum_{x \in \mathcal{L}} \left(\frac{1}{3} \text{Tr}(\Phi^3(x)) - \sum_{\alpha \in I} \text{Tr}(\Phi(x) \Phi_\alpha^2(x)) + \frac{2}{3} \sum_{\alpha \in I} \text{Tr}(\Phi_\alpha^3(x)) \right) \quad , \quad (70)$$

where:

$$\Phi_\alpha(x) = \frac{1}{1-\lambda'} (\mathbf{A}'_\alpha \Phi(x + V(\alpha)) \mathbf{A}'_{\bar{\alpha}} - \lambda' \Phi(x)) = \quad (71)$$

$$\sum_{y \in \mathcal{L}} \sum_{l=1}^{\infty} \sum_{\hat{\gamma} \in \bar{\mathcal{S}}_l^\alpha(x \rightarrow y)} A'(\hat{\gamma}) J'(y) A'(\hat{\gamma}^{-1}) \quad . \quad (72)$$

The effective action $\Gamma_0(\Phi)$ can be obtained by a Legendre transformation from $W_0(J)$. The cubic term is precisely given by minus the corresponding cubic term in $W_0(J)$, displayed in Eq. (70). Following a similar procedure one can use the formulas of the previous sections to compute quartic and higher terms in $\Gamma_0(\Phi)$.

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References

- [1] K.G. Wilson. *New Phenomena in subnuclear physics* , Erice Lectures 1975, ed A. Zichichi (Plenum 1977).
- [2] M. Creutz, *Quarks, Gluons and Lattices*. Cambridge University Press, Cambridge, 1983.
H.J. Rothe, *Lattice Gauge Theories. An Introduction*. World Scientific, Singapore, 1992.
- [3] N. Kawamoto, *Nucl. Phys.* **B190[FS3]** (1981) 617.
- [4] J. Fröhlich and C. King, *Nucl. Phys.* **B290[FS20]** (1987) 157.
- [5] E. Brezin and D.J. Gross, *Phys. Lett.* **97B** (1980) 120;
H. Kluberg-Stern, A. Morel, O. Napoly and B. Petersson, *Nucl. Phys.* **B190[FS3]** (1981) 504.
N. Kawamoto and J. Smit, *Nucl. Phys.* **B192** (1981) 100.
S. Aoki, *Phys. Rev.* **D30** (1984) 2653.
S. Aoki, *Phys. Rev.* **D33** (1986) 2399.
- [6] E. Gabrielli, A. González-Arroyo and C. Pena. N=1 Supersymmetric Yang-Mills on the lattice at strong coupling. preprint FTUAM-98-24, IFT-UAM/CSIC-99-4; hep-th/9902209.
- [7] A. González-Arroyo and C. Pena, *SU(N) group integration for gauge fields in the adjoint representation at large N*, preprint FTUAM-99-5, IFT-UAM/CSIC-99-6.